# Tannaka-Krein Duality for Association Schemes 

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#### Abstract

A duality theorem is formulated for noncommutative association schemes. This duality theorem contains as special cases (1) the Delsarte-Tamaschke duality theorem (which was essentially obtained by Kawada in 1942) for commutative association schemes, and (2) the Tannaka-Krein duality theorem for arbitrary finite groups.


## INTRODUCTION

An association scheme of class $d$ is a pair $\chi=(X, \Re)$ consisting of a finite set $X$ of $n$ points and a set $\Re=\left\{R_{0}, R_{1}, \ldots, R_{d}\right\}$ of relations $R_{i} \neq \varnothing$ which satisfy:
(1) $R_{0}=\{(x, x) \mid x \in X\}$ is the identity relation.
(2) For every $x, y \in X,(x, y) \in R_{i}$ for exactly one $i$.
(3) For each $i \in\{0,1, \ldots, d\},{ }^{t} R_{i}\left[:=\left\{(y, x) \mid(x, y) \in R_{i}\right\}\right]=R_{i}$ for some $j$.
(4) For each $i, j, k \in\{0,1, \ldots, d\},\left|\left\{z \in X \mid(x, z) \in R_{i},(y, z) \in R_{i}\right\}\right|$ is constant ( $p_{i k}^{i}$ ) whenever $(x, y) \in R_{k}$.

In some literature the definition of association scheme is slightly different. In [4], $p_{j k}^{i}=p_{i k}^{i}$ is assumed, and in [2] more strongly ${ }^{t} R_{i}=R_{i}$ is assumed. The present definition is equivalent to a homogeneous coherent configuration in the sense of Higman [7].

Let $A_{i}(i=0,1, \ldots, d)$ be the adjacency ( $n$ by $n$ ) matrix with respect to the relation $R_{i}$. Then the algebra $\mathfrak{A}=\left\langle A_{0}, A_{1}, \ldots, A_{d}\right\rangle$ spanned by the $A_{i}$ over $C$ is an algebra of dimension $d+1$, and is called Bose-Mesner algebra or Hecke algebra. The case when the algebra $\mathfrak{A l}$ is commutative has already been extensively studied. If $\mathfrak{A}$ is commutative, then it is well known that $\mathfrak{A}$ has a unique set of primitive idempotents $E_{0}, E_{1}, \ldots, E_{d}$ that also becomes a basis of the algebra $\mathfrak{M}$, and a duality theorem between the two bases $\left\{A_{i}\right\}$ and $\left\{E_{i}\right\}$ is
well known. (Cf. [4], [11], [8].) Note that in terms of Delsarte [4] this is a duality between the ordinary multiplication and the Hadamard multiplication (i.e., pointwise multiplication) in the algebra $\mathscr{H}$, which is a subalgebra of the full matrix algebra. In other words, the $A_{i}$ are characterized as the unique set of primitive idempotents in the algebra $\mathfrak{U}$ whose multiplication is defined by the Hadamard multiplication. It is also interesting to note that this duality was obtained as early as 1942 by Kawada [8] in a deeper way. Namely, the existence of an association scheme was not necessary, and this duality was formulated for those algebras (called C-algebras by Kawada) whose property was extracted from that of the Bose-Mesner algebra. We will refer this duality as Delsarte-Tamaschke-Kawada duality in what follows.

Now it seems natural and interesting to ask what happens if the algebra $\mathfrak{A}$ is not commutative. The purpose of this paper is to answer this question.

Krein [9] (see also [6, §30]) interpreted the Tannaka duality (for a compact group) more precisely, by introducing a dual object, which is now called the Krein algebra. In this paper we will follow the treatment by Krein, and we will construct an object similar to the Krein algebra as a dual object for the algebra $\mathfrak{A}$. Then we recover the structure of algebra $\mathfrak{A}$ by taking the multiplicative functionals of the new algebra. This mechanism is very similar (and almost identical) to the proof of Tannaka-Krein duality by Krein [9] (see [6]), and also the calculations involved here are essentially already known (see [11], [7]). However, I hope that this formulation is new and that it makes the meaning of Tannaka-Krein duality more clear.

This new duality contains both Delsarte-Tamaschke-Kawada duality (for commutative association schemes) and Tannaka-Krein duality (for finite groups) as special cases. The first one is obtained by adding the extra assumption of commutativity, and the second one corresponds to our duality for the algebra attached to the double coset space $H \backslash G / H$ when we specialize to $H=1$.

It would be interesting to know whether this duality is generalized to nonfinite case, in particular for $H \backslash G / H$ with $G$ any compact topological group and $H$ any closed subgroup of $G$.

The content of this paper was presented as a part of my talk at the AMS San Francisco meeting in January 1981 entitled "McKay's observation, Delsarte's theory on association schemes, and a duality theorem of the character table of a finite group" (Abstract 783-05-57; see also [1]). During and after the talk, several people pointed out that similar results had recently been discussed in harmonic analysis ([5], [10], etc.). I have checked those references, and so far have been able to find only those results corresponding to Delsarte-Tamaschke-Kawada duality, i.e. the commutative case, ignoring measure theoretical complications. However, I think that it would have been easier for experts in harmonic analysis to formulate the duality presented here
(if they wished), and that the reason why they did not mention this duality is perhaps that they are rather interested in general results for $H \backslash G / H$ for any compact group $G$, but the situation is quite difficult and unclear if $G$ is not finite. Anyway, this connection has made me aware that the concept of hypergroup in harmonic analysis is essentially the same as that of association scheme (when we ignore measure theoretical complications), and that there are close connections between these two theories.

## 1. DUALITY THEOREM

Let $\chi=\left(X,\left\{R_{i}\right\}\right)$ be a noncommutative (i.e., not necessarily commutative) association scheme of class $d$, and let $\mathfrak{A}=\left\langle A_{0}, A_{1}, \ldots, A_{d}\right\rangle$ be the Bose-Mesner algebra, a subalgebra of $M_{n}(C)$. Since $\mathfrak{A}$ is closed under the conjugate-transpose map, $\mathfrak{A}$ has no nontrivial left nilpotent ideal, and so $\mathfrak{A}$ is a semisimple algebra. Thus $\mathfrak{H}$ is a direct sum of complete matrix algebras $C_{i}\left[\cong M_{e_{i}}(C)\right]$ over $C$ :

$$
\mathfrak{A}=C_{0} \oplus C_{1} \oplus \cdots \oplus C_{m}
$$

with $d+\mathrm{I}=\sum_{i=0}^{m} e_{i}^{2}$. The algebra $\mathfrak{A}$ is completely reducible, that is, there is a nonsingular matrix $U \in M_{n}(C)$, and even a unitary one if we need that, such that

$$
U^{-1} \phi U=\operatorname{diag}(\underbrace{\Delta_{0}(\phi)}_{z_{0}=1}, \underbrace{\Delta_{1}(\phi), \ldots, \Delta_{1}(\phi)}_{z_{1}}, \ldots, \underbrace{\Delta_{m}(\phi), \ldots, \Delta_{m}(\phi)}_{z_{m}})
$$

for all $\phi \in \mathfrak{A}$, where $\Delta_{0}, \Delta_{1}, \ldots, \Delta_{m}$ are inequivalent irreducible representations of $\mathfrak{A}$ such that the degree of $\Delta_{i}$ is $e_{i}$. So we have $n=\sum_{i=0}^{m} e_{i} z_{i}$. We write

$$
\Delta_{p}(\phi)=\left(a_{i j}^{\nu}(\phi)\right)
$$

Since $\left\{A_{0}, A_{1}, \ldots, A_{d}\right\}$ is a basis of $\mathfrak{H}$, the function $a_{i j}^{v}$ on $\mathfrak{H}$ is determined by the values of the $a_{i j}^{\nu}\left(A_{k}\right)$. Let us set $Y=\left\{A_{0}, A_{1}, \ldots, A_{d}\right\}$, and let $\mathfrak{A}^{\#}$ be the set of all ( $C$-valued) functions on $Y$ generated by the $a_{i j}^{\nu}$ (with $0 \leqslant \nu \leqslant m$, and $1 \leqslant i \leqslant e_{\nu}, \mathbf{1} \leqslant i \leqslant e_{\nu}$ ). Then by a theorem of Frobenius and Schur [3, (27.3)] the $a_{i j}^{\nu}$ are linearly independent on $Y$, and so $\mathfrak{A}^{\#}$ is the set of all functions on $Y$ (or see Lemma 1 below). The $\mathfrak{A}^{\#}$ has a natural multiplication operation, namely pointwise multiplication, and $\mathfrak{I}^{\#}$ becomes a commutative algebra with respect to this pointwise multiplication. The algebra $\mathfrak{A}^{\#}$ (with the
specified basis $a_{i j}^{\nu}$ ) plays the same role as the Krein algebra does in TannakaKrein duality (cf. [6, §30], [9]). For an algebra, say A, a linear function from $A$ to $C$ is said to be a linear multiplicative functional if

$$
f(x y)=f(x) f(y)
$$

for all $x$ and $y$ in $A$. For an algebra $A$, let $H_{A}$ be the set of all linear multiplicative functionals of $A$.

Now our duality theorem is formulated as follows.

Theorem 1. Let $\mathfrak{A}=\left\langle A_{0}, A_{1}, \ldots, A_{d}\right\rangle$ be the Bose-Mesner algebra for an association scheme of class $d$. Let $\mathfrak{Q}^{\#}$ with the specified basis $a_{i j}^{\nu}$ $\left(0 \leqslant \nu \leqslant m\right.$ and $1 \leqslant i \leqslant e_{\nu}$ and $\left.1 \leqslant j \leqslant e_{v}\right)$ be the above-defined algebra for $\mathfrak{A}$. Let $H_{\mathfrak{A}}{ }^{*}$ be the set of all linear multiplicative functionals of $\mathfrak{A}^{\#}$. Then $H_{\mathfrak{A}^{\mp}}$ is identified with the set $Y=\left\{A_{0}, A_{1} \cdots A_{d}\right\}$. For $\varphi$ and $\psi$ in $H_{\mathfrak{A}^{\mp}}$ let us define the product $\varphi \psi$ by

$$
\begin{equation*}
\varphi \psi\left(a_{i j}^{v}\right)=\sum_{r=0}^{e_{v}} \varphi\left(a_{i r}^{v}\right) \psi\left(a_{r i}^{v}\right) . \tag{1.1}
\end{equation*}
$$

Then $\varphi \psi$ is a linear combination of elements in $H_{\mathfrak{A}^{*}}$, and this algebra structure on $H_{\mathfrak{U}^{+}}$is isomorphic with that of the original Bose-Mesner algebra $\mathfrak{A}$. Precisely speaking, if $A_{\varphi}, A_{\psi}$, and $A_{\mu}$ are the elements in $Y$ corresponding to $\varphi, \psi$, and $\mu$ respectively, then

$$
\begin{equation*}
A_{\varphi} A_{\psi}=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} A_{\mu} \tag{1.2}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
\varphi \psi=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} \mu \tag{1.3}
\end{equation*}
$$

## 2. PKOOF OF THEOREM 1

1. 

First we show that $H_{M{ }^{\#}}$ is identified with the set $Y$. As we mentioned before, $\mathscr{U}^{\#}$ is the set of all functions on $Y$, and so we have clearly $\mathfrak{U}^{\#}=C_{0}(Y)$,
because $Y$ is a finite set. Therefore it is known that the structure space (i.e. $\left.H_{9{ }^{\#}}\right)$ is identified with the set of evaluation functions by an element in $Y$. (The reader is referred to Appendix C in [6], in particular (C.29), (C.30), and (C.32) on pp. $482-483$ in [6, Part I].)
2.

To prove the second part, we have only to show that the relation (1.2) implies the relation (1.3). That is to say, we have only to show that

$$
\begin{equation*}
\varphi \psi\left(a_{i j}^{\tau}\right)=\sum_{\mu=0}^{d} p_{\psi \mu}^{\psi} \mu\left(a_{i j}^{\tau}\right) \tag{2.1}
\end{equation*}
$$

for all $a_{i j}^{\tau}$.
Now we introduce the following notation. Let

$$
\mathrm{Z}=\left[\begin{array}{cccc}
z_{0} E_{e_{0}^{2}} & & & \bigcirc \\
& z_{1} E_{e_{1}^{2}} & & \\
& & \ddots & \\
\bigcirc & & & z_{m} E_{e_{m}^{2}}
\end{array}\right], \quad T=\left[\begin{array}{cccc}
k_{0} & & & \bigcirc \\
& k_{1} & & \\
& & \ddots & \\
\bigcirc & & & k_{d}
\end{array}\right]
$$

(where $E_{l}$ is the identity matrix of size $l$ ) with $k_{i}=p_{i 0}^{i}$. Also, set

$$
F=\left(a_{\alpha \beta}^{(\nu)}\left(A_{i}\right)\right)_{(\nu, \alpha, \beta), i}
$$

and

$$
F^{*}=\left(a_{\alpha \beta}^{(\nu)}\left(A_{i^{\prime}}\right)\right)_{(\nu, \alpha, \beta), i}
$$

where the triples $(\nu, \alpha, \beta)$ with $\alpha, \beta=1, \ldots, e_{\nu}, \nu=0,1, \ldots, m$ in their lexicographic order are used as row indices, and the $i=0,1, \ldots, d$ as column indices. Then we have the following:

Lemma 1 (Schur relations).
(i) $F^{\prime} Z F^{*}=n T$.
(ii) $F^{*} Z F^{\prime}=n T$.

Proof. In the special case when the algebra $\mathfrak{N}$ is the center of the group ring of a finite group, these relations are the orthogonality relations of irreducible characters written in a slightly different form from usual.

These relations are well known and called Schur relations. The reader is referred to [7, 11]. Note that (i) is (3.12) in [7] (and essentially the same as (2.3) in [11]), and (ii) is (3.11) in [7] (and essentially the same as (2.4) in [11]).

Lemma 2. Let

$$
\begin{equation*}
A_{\varphi} A_{\psi}=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} A_{\mu} . \tag{1.2}
\end{equation*}
$$

Then the structure constant $p_{\psi_{\mu}}^{\varphi}$ is given by

$$
\begin{equation*}
p_{\psi \mu}^{\varphi}=\frac{1}{n k_{\mu}} \sum_{\nu=0}^{m} z_{v} \sum_{\alpha, \beta, \gamma=1}^{e_{\nu}} a_{\alpha \beta}^{(\nu)}\left(A_{\varphi}\right) a_{\beta \gamma}^{(\nu)}\left(A_{\psi}\right) a_{\gamma \alpha}^{(\nu)}\left(A_{\mu^{\prime}}\right) . \tag{2.2}
\end{equation*}
$$

Proof. Multiplying (1.2) by $A_{\mu}$, we have

$$
\begin{equation*}
A_{\varphi} A_{\psi} A_{\mu^{\prime}}=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} A_{\mu} A_{\mu^{\prime}} \tag{2.3}
\end{equation*}
$$

Now apply $z_{\nu} f_{\alpha \alpha}$ to both sides of (2.3). This is essentially the same as taking the trace of (2.3). Then applying Lemma 1(ii), we get the desired result. (This calculation is essentially done in [11, p. 299].)

## 3. Completion of the Proof of Theorem 1

To complete the proof of Theorem 1, we have only to show that

$$
\varphi \psi\left(a_{i j}^{\tau}\right)=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} \mu\left(a_{i j}^{\tau}\right)
$$

for all $a_{i j}^{\tau}$. That is, we have only to show that

$$
\begin{equation*}
\sum_{\mathrm{r}=0}^{e_{\tau}} \varphi\left(a_{i r}^{\tau}\right) \psi\left(a_{r i}^{\tau}\right)=\sum_{\mu=0}^{d} p_{\psi \mu}^{\varphi} \mu\left(a_{i j}^{\tau}\right) . \tag{2.4}
\end{equation*}
$$

This is immediately verified by using Lemma 2 and Lemma 1 (ii), because

$$
\begin{equation*}
\sum_{\nu=0}^{m} z_{\nu} f_{\gamma \alpha}^{(\nu)}\left(A_{\mu^{\prime}}\right) f_{i j}^{(\tau)}\left(A_{\mu}\right)=\left(n \cdot k_{\mu}\right) \delta_{(\nu, \gamma, \alpha),(\tau, i, i)} \tag{2.5}
\end{equation*}
$$

by Lemma 1(ii).
Thus the proof of Theorem 1 has been completed.

## 3. REMARKS

(i) It is easy to see that if $\mathfrak{H}$ is commutative, then all $e_{\nu}=1$ and we get the Delsarte-Tamaschke-Kawada duality.
(ii) Let $G$ be a finite group, and let $H$ be a subgroup. Then as is well known, the set of double cosets $H \backslash G / H$ has the structure of a (not necessary commutative) association scheme. It is clear that the original Tannaka-Krein duality is essentially our duality for $H \backslash G / H$ when we specialize to $H=1$. [An important feature of the original Tannaka-Krein duality is that the product $\varphi \psi$ in our (1.3) becomes a single element in $H_{9^{\text {. }}}$ ]
(iii) As Kawada [8] proved his duality theorem for C -algebras (in his terminology), our duality theorem can also be formulated for algebras which do not necessarily come from an association scheme.
(iv) As we mentioned in the Introduction, it would be interesting to know whether we can obtain similar duality theorems to ours for $H \backslash G / H$ with a nonfinite compact topological group $G$ and a closed subgroup $H$.

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